

Section 7.4. Integration of Rational functions by Partial Fractions

Goal: integrate rational functions $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials.

- The degree of a polynomial is the highest power in that polynomial.
- We deal with "complicated" rational functions by re-writing them as a sum (or difference) of "simpler" rational functions. For example, the function $f(x) = \frac{x+5}{x^2+x-2}$ may be written as $f(x) = \frac{2}{x-1} - \frac{1}{x+2}$. Note that it is much easier to integrate the function $f(x) = \frac{2}{x-1} - \frac{1}{x+2}$ than to integrate the original, equivalent function $f(x) = \frac{x+5}{x^2+x-2}$.
- Step Zero: If degree of P \geq degree of Q, start with long division.

Example ① Find the anti-derivative $\int \frac{x^3+x-1}{x-1} dx$.

$$\begin{array}{r} x^3 + x - 1 \\ -(x^3 - x^2) \\ \hline x^2 + x - 1 \\ -(x^2 - x) \\ \hline 2x - 1 \\ -(2x - 2) \\ \hline 1 \end{array} \quad \left| \begin{array}{l} x-1 \\ x^2+x+2 \end{array} \right.$$

Long division gives us

$$\frac{x^3+x-1}{x-1} = x^2+x+2 + \frac{1}{x-1}$$

We will integrate the much simpler function on the right

$$\int x^2+x+2 + \frac{1}{x-1} dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + \ln|x-1| + C$$

- Step 1: factor the denominator Q(x) completely, into a product of "linear factors" ($ax+b$) and/or "irreducible quadratic factors" (x^2+a^2). For example, if $Q(x) = x^4-81$, we write $Q(x) = (x^2-9)(x^2+9) = (x-3)(x+3)(x^2+9)$. Here $(x+3)$ and $(x-3)$ are linear factors, and (x^2+9) is an irreducible quadratic factor.

• Step 2: Setup of partial fractions. This step has many cases:

- Case I: $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x+b_1)(a_2x+b_2) \cdots (a_nx+b_n);$$

In this case we write $\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$, and solve for the A_i 's.

Example ② Find the antiderivative $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} dx$.

• Long division is not needed (degree of $P <$ degree of Q).

• We factor $Q(x)$ as $Q(x) = x(2x^2+3x-2) = x(2x-1)(x+2)$. So, $Q(x)$ is a product of 3 distinct linear factors. we setup partial fraction

$$\text{as } \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \quad (*)$$

To solve for A, B and C , we first multiply both sides of $(*)$ by $Q(x)$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \quad (\Delta)$$

Now, (Δ) holds for all values of x ; specifically, it holds for the "special values" $x=0$, $x=\frac{1}{2}$, and $x=-2$. Let's plug these values in.

$$x=0; \quad 0^2+2\cdot 0-1 = A(-1)(2) + B\cdot 0\cdot(0+2) + C\cdot 0\cdot(-1) \Rightarrow A = \frac{1}{2}.$$

$$x=\frac{1}{2}; \quad \frac{1}{4}+1-1 = A\cdot 0\cdot\left(\frac{5}{2}\right) + B\cdot\frac{1}{2}\cdot\frac{5}{2} + C\cdot\frac{1}{2}\cdot 0 \Rightarrow B = \frac{1}{5}.$$

$$x=-2; \quad 4-4-1 = A\cdot(-5)(0) + B(-2)\cdot 0 + C\cdot(-2)(-5) \Rightarrow C = -\frac{1}{10}.$$

Therefore, we have $\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{1}{2}\cdot\frac{1}{x} + \frac{1}{5}\cdot\frac{1}{2x-1} - \frac{1}{10}\cdot\frac{1}{x+2}$.

Finally, the antiderivative is $\frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$.

- Case II: $Q(x)$ is a product of linear factors, some of which are repeated.

For example, $Q(x) = x^2(x-1)^3$ is a product of two linear factors:

x , repeated twice, and $(x-1)$, repeated three times. We say (x) has multiplicity 2 and $(x-1)$ has multiplicity 3. The partial fraction setup of

$$\frac{P(x)}{x^2(x-1)^3} \text{ is } \frac{P(x)}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}.$$

As you can see, we have to account for the multiplicity of each linear factor.

Example ③ Find the anti-derivative $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

Step Zero: long division yields an equivalent function: $x+1 + \frac{4x}{x^3 - x^2 - x + 1}$

Step 1: $Q(x) = x^3 - x^2 - x + 1$ factors as $(x-1)(x-1)(x+1) = (x+1) \cdot (x-1)^2$.

Step 2: the linear factor $(x+1)$ has multiplicity 1, and the linear factor $(x-1)$ has multiplicity 2. We then have the following partial fractions:

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x-1)^2} + \frac{C}{x-1}. \text{ Multiply both sides by } Q(x) \text{ to get:}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2. \text{ Plug-in the values } x=1 \text{ and } x=-1$$

$$x=1; \quad 4 = B \cdot 2 \Rightarrow B = 2$$

$$x=-1; \quad -4 = C(-2)^2 \Rightarrow C = -1. \text{ Choose any other value for } x \text{ to get } A.$$

$$x=0; \quad 0 = -A + B + C \Rightarrow A = B + C = -1 + 2 = 1. \text{ Therefore we have}$$

$$\begin{aligned} \int x+1 + \frac{4x}{(x+1)(x-1)^2} dx &= \int x+1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C \end{aligned}$$

- Case III: $Q(x)$ contains irreducible quadratic factors, none of which are repeated. For example, $Q(x) = (x-2)(x^2+1)(x^2+4)$ contains a linear factor $(x-2)$, and two irreducible quadratic factors (x^2+1) and (x^2+4) , each of multiplicity 1.

When setting up partial fractions for this case, we correspond to each of the quadratic factors, a linear numerator of the form $(mx+b)$.

For example, the partial fractions setup for $\frac{x}{(x-2)(x^2+1)(x^2+4)}$ is given by

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}.$$

Example ④ Find the anti-derivative $\int \frac{2x^2-x+4}{x^3+4x} dx$.

Long division is not needed, and $Q(x) = x^3+4x = x(x^2+4)$ is a product of a linear factor (x) and an irreducible quadratic factor (x^2+4) , each of multiplicity 1. Therefore, we have the partial fractions

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}. \text{ Multiply both sides by } Q(x) \text{ to get}$$

$$2x^2-x+4 = A(x^2+4) + (Bx+C) \cdot x \quad (\nabla)$$

with irreducible quadratic terms, we don't have many "nice" values for x . Let us use a different method to solve for A, B , and C than before. The method is called "Comparison of Coefficients". Start by expanding the right-hand-side of (∇) ; you get

$$2x^2-x+4 = Ax^2 + 4A + Bx^2 + CX; \text{ now combine alike terms:}$$

$2x^2-x+4 = (A+B)x^2 + Cx + 4A$; these two polynomials are equal when their corresponding coefficients are equal. In other words,

$$\left. \begin{array}{l} A+B=2 \\ C=-1 \\ 4A=4 \end{array} \right\} \Rightarrow \left. \begin{array}{l} A=1 \\ B=1 \\ C=-1 \end{array} \right\} \text{ and so } \frac{2x^2-x+4}{x(x^2+4)} = \frac{1}{x} + \frac{x-1}{x^2+4}. \text{ We now find}$$

$$\begin{aligned} \int \frac{1}{x} + \frac{x-1}{x^2+4} dx &= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4} dx \\ &= \ln|x| + \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C. \end{aligned}$$

In some cases, integrals that might seem to require partial fractions actually don't. Here's an example:

Example ⑤ Find the anti-derivative $\int \frac{x-1}{4x^2-4x+3} dx$.

No long division is necessary, and $Q(x) = 4x^2-4x+3$ does not factor! try instead Completing the square on Q :

$$Q(x) = 4x^2-4x+3 = (2x-1)^2 + 2 \Rightarrow \frac{x-1}{4x^2-4x+3} = \frac{x-1}{(2x-1)^2 + 2}.$$

$$\text{Now, } \int \frac{x-1}{(2x-1)^2 + 2} dx = \int \frac{\frac{1}{2}(2x-1) - \frac{1}{2}}{(2x-1)^2 + 2} dx = \frac{1}{2} \int \frac{2x-1}{(2x-1)^2 + 2} dx - \frac{1}{2} \int \frac{1}{(2x-1)^2 + 2} dx$$

For the first integral, let $u = (2x-1)^2$; then $du = 2(2x-1) \cdot 2 dx$
so that $(2x-1) dx = \frac{1}{4} du$.

For the second integral, let $V = 2x-1$; then $dv = 2dx \rightarrow dx = \frac{1}{2} dv$.

$$\text{Therefore we have } \frac{1}{2} \int \frac{1}{4} \frac{du}{u+4} - \frac{1}{2} \int \frac{1}{2} \frac{dv}{v^2+2}$$

$$= \frac{1}{8} \ln|u+4| - \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{V}{\sqrt{2}}\right) + C = \frac{1}{8} \ln|(2x-1)^2 + 4| - \frac{1}{4\sqrt{2}} \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C.$$

- Case IV: $Q(x)$ Contains a repeated irreducible quadratic factor. (Similar to Case II)

$$\text{Example } \frac{x^3+x-1}{x(x+1)^2(x^2+4)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{x^2+1} + \frac{Hx+I}{(x^2+1)^2} + \frac{Jx+K}{(x^2+1)^3}.$$

This example contains all the previous cases!